ARITHMETICAL RANK OF THE CYCLIC AND BICYCLIC GRAPHS

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ABSTRACT. We show that for the edge ideals of the graphs consisting of one cycle or two cycles of any length connected through a vertex or a path, the arithmetical rank equals the projective dimension.

1. INTRODUCTION

For any homogeneous ideal I of a polynomial ring $R = K[x_1, \ldots, x_n]$ there exists a *graded* minimal finite free resolution

$$0 \to \bigoplus_j R(-j)^{\beta_{pj}} \to \ldots \to \bigoplus_j R(-j)^{\beta_{1j}} \to R \to R/I \to 0$$

of R/I, in which $R(-j)$ denotes the graded free module obtained by shifting the degrees of elements in R by j. The numbers β_{ij}, which we shall refer to as the ith Betti numbers of degree j of R/I, are independent of the choice of the graded minimal finite free resolution. We also define the ith Betti number of I as $\beta_i := \sum \beta_{ij}$.

Given a polynomial ring R over a field, and a graph G having the set of indeterminates as its vertex set $V(G)$ and the set of edges $E(G)$, one can associate with G a monomial ideal of R: this ideal is generated by the products of the vertices of each edge in $E(G)$, and is hence generated by squarefree quadratic monomials. It is called the *edge ideal* $I(G)$ of G, and has been intensively studied by Simis, Vasconcelos and Villarreal in [17]. The arithmetical rank (ara), i.e., the least number of elements of R which generate a given monomial ideal up to radical, is in general bounded below by its projective dimension (pd), i.e., by the length of every minimal free resolution of the quotient of R with respect to the ideal. The simplicial complex Δ_G of a graph G is defined by

$$\Delta_G = \{A \subseteq V(G) | A \text{ is an independent set in } G\},$$

where A is an independent set in G if none of its elements are adjacent. Note that Δ_G is precisely the Stanley-Reisner simplicial complex of $I(G)$.

For any simplicial complex Δ on the vertex set $V(\Delta)$, the *Alexander dual* of Δ is the simplicial complex defined by

$$\Delta^* := \{F \subseteq V(\Delta) | \ V(\Delta) \setminus F \notin \Delta\}.$$

The link of a face $F \in \Delta$ is defined as the simplicial complex

$$Link_\Delta F := \{G \in \Delta | G \cup F \in \Delta \ and \ G \cap F = \emptyset\}.$$

In recent times, the projective dimension has been determined for large classes of edge ideals, where it is independent of the ground field: in Jacques' thesis it was computed for acyclic graphs (see also [12]), but also for the graphs C_n, consisting of one cycle of length n. Jacques, in [11, Theorem 6.1.8], using Hochster's formula [9], showed that for a graph G, the Betti numbers are

(*) $$\beta_{i,d}(G) = \sum_{H \subset G, |V(H)|=d} dim_k \tilde{H}_{i-2}(\varepsilon(H); K).$$

Then he used (*) for providing formulas for the graded Betti numbers of special classes of graphs including lines, cycles and complete graphs. He proved the following theorems.

Theorem A [11, Lemma 8.2.7] The reduced homology of the disjoint union of the cyclic graph C_n and any non empty graph G may be expressed as follows:

$$\tilde{H}_i(\varepsilon(C_n \cup G); K) = \begin{cases} \tilde{H}_{i-\frac{2n+1}{3}}(\varepsilon(G); K), & \text{if } n \equiv 1 \bmod 3 \\ \tilde{H}_{i-\frac{2n-1}{3}}(\varepsilon(G); K), & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

Theorem B [11, Corollary 7.6.30] The non zero Betti numbers in degree n and the projective dimension of C_n in degree n are the following:

$\beta_{\frac{2n}{3},n} = 2,$ and $pd \ I(C_n) = \frac{2n}{3}$, if $n \equiv 0 \bmod 3$,
$\beta_{\frac{2n+1}{3},n} = 1,$ and $pd \ I(C_n) = \frac{2n+1}{3}$, if $n \equiv 1 \bmod 3$,
$\beta_{\frac{2n-1}{3},n} = 1,$ and $pd \ I(C_n) = \frac{2n-1}{3}$, if $n \equiv 2 \bmod 3$.

Theorem C [11, Lemma 8.1.3] The reduced homology of the disjoint union of the line graph L_n and any non empty graph G may be expressed as follows:

$$\tilde{H}_i(\varepsilon(L_n \cup G); K) = \begin{cases} \tilde{H}_{i-\frac{2n}{3}}(\varepsilon(G); K), & \text{if } n \equiv 0 \bmod 3 \\ 0, & \text{if } n \equiv 1 \bmod 3 \\ \tilde{H}_{i-\frac{2n-1}{3}}(\varepsilon(G); K), & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

From the proof of [11, Corollary 7.7.35] one can derive the following result.

Theorem D [11, Corollary 7.7.35] The projective dimension of the line graph is independent of the characteristic of the chosen field and is

$$pd \ I(L_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \bmod 3 \\ \frac{2n-2}{3} & \text{if } n \equiv 1 \bmod 3 \\ \frac{2n-1}{3} & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

All Betti numbers of L_n in degree n are zero if $n \equiv 1 \bmod 3$. Otherwise the non zero Betti numbers of degree n of L_n are

$$\beta_{\frac{2n}{3},n} I(L_n) = 1, \text{if } n \equiv 0 \bmod 3,$$

$$\beta_{\frac{2n-1}{3},n} I(L_n) = 1, \text{if } n \equiv 2 \bmod 3.$$

In [6] an explicit formula is given for the Betti numbers of a special kind of bipartite graphs, the so-called *Ferrers graphs*. In [2] it is shown that the arithmetical rank equals the projective dimension for a special class of acyclic graphs, in [3] that

this is also true for all Ferrers graphs. In the present paper we prove that the same equality holds for all cyclic and bicyclic graphs. By *bicyclic graph* we mean a graph which consists of two cycles that have exactly one vertex in common or are connected by a path. In particular, we will see that the projective dimension of the edge ideals of these graphs does not depend on the characteristic of the ground field.

2. The arithmetical rank of cyclic graphs

Let K be a field, and consider the polynomial ring $R = K[x_1, \ldots, x_n]$, where $n \geq 3$. Let C_n be the graph on the vertex set $\{x_1, \ldots, x_n\}$ whose set of edges is $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_1, x_n\}\}$. Then its edge ideal is the following ideal of R:

$$I(C_n) = (x_1 x_2, x_2 x_3, \ldots, x_{n-1} x_n, x_1 x_n).$$

We will show that for all n, $pd\, I(C_n) = ara\, I(C_n)$. In general, for any ideal I of R we have that $cd\, I \leq ara\, I$, where cd denotes the local cohomological dimension (see [8], Example 2, p. 414) and, whenever I is a monomial ideal, $pd\, I = cd\, I$ (see [14], Theorem 1). Hence it will suffice to show that, for all n, $ara\, I(C_n) \leq pd\, I(C_n)$, i.e., to produce $pd\, I(C_n)$ elements of R generating $I(C_n)$, up to radical. Among the available tools, we have, on the one hand, Jacques' result providing explicit formulas for the projective dimension of $I(C_n)$.

On the other hand, we know that a finite set of elements of R which generate a given ideal up to radical can be constructed according to the following criterion, which is due to Schmitt and Vogel.

Lemma 2.1. ([16], p. 249) *Let P be a finite subset of elements of R. Let $P_0, \ldots, P_r$ be subsets of P such that*

    (i) $\bigcup_{i=0}^{r} P_i = P$;
    (ii) P_0 has exactly one element;
    (iii) if p and p' are different elements of P_i $(0 < i \leq r)$ there is an integer
        i' with $0 \leq i' < i$ and an element in $P_{i'}$ which divides pp'.

We set $q_i = \sum_{p \in P_i} p^{e(p)}$, where $e(p) \geq 1$ are arbitrary integers. We will write (P) for the ideal of R generated by the elements of P. Then we get

$$\sqrt{(P)} = \sqrt{(q_0, \ldots, q_r)}.$$

We have to distinguish between three cases, depending on the residue of n modulo 3. The cases $n \equiv 0, 1 \mod 3$ can be settled by a direct application of Lemma 2.1. The case $n \equiv 2 \mod 3$ is more interesting, since it needs some additional non trivial computations on the generators.

Proposition 2.2. *Suppose that $n = 3m$, for some integer m. Set $q_0 = x_1 x_2$, $q_1 = x_1 x_{3m} + x_2 x_3$, and, for $1 \leq i \leq m - 1$, set*

$$q_{2i} = x_{3i+1} x_{3i+2}$$
$$q_{2i+1} = x_{3i} x_{3i+1} + x_{3i+2} x_{3i+3}.$$

Then

$$I(C_n) = \sqrt{(q_0, \ldots, q_{2m-1})}.$$

In particular, $ara\, I(C_n) \leq 2m$.

Proof. For all $i = 0, \ldots, m - 1$, the monomial q_{2i} divides the product of the two summands of q_{2i+1}. By Lemma 2.1 it follows that

$$(x_{3i}x_{3i+1}, x_{3i+1}x_{3i+2}, x_{3i+2}x_{3i+3}) = \sqrt{(q_{2i}, q_{2i+1})}.$$

This implies the claim. $\qquad\square$

Using the same arguments as in the proof of Proposition 2.2, from Lemma 2.1 we can deduce the next result.

Proposition 2.3. *Suppose that* $n = 3m + 1$, *for some integer* m. *Set* $q_0 = x_1x_2$, $q_1 = x_1x_{3m+1} + x_2x_3$, *and, for* $1 \le i \le m - 1$, *set*

$$q_{2i} \quad = \quad x_{3i+1}x_{3i+2}$$
$$q_{2i+1} \quad = \quad x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3},$$

and, finally, $q_{2m} = x_{3m}x_{3m+1}$. *Then*

$$I(C_n) = \sqrt{(q_0, \ldots, q_{2m})}.$$

In particular, ara $I(C_n) \le 2m + 1$.

Proposition 2.4. *Suppose that* $n = 3m + 2$, *for some integer* m. *Set* $q_0 = x_1x_2$, $q_1 = x_2x_3 + x_4x_5$, *and, for* $1 \le i \le m - 1$, *set*

$$q_{2i} \quad = \quad x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3}$$
$$q_{2i+1} \quad = \quad x_{3i+2}x_{3i+3} + x_{3i+4}x_{3i+5},$$

and, finally, $q_{2m} = x_1x_{3m+2} + x_{3m}x_{3m+1}$. *Then*

$$I(C_n) = \sqrt{(q_0, \ldots, q_{2m})}.$$

In particular, ara $I(C_n) \le 2m + 1$.

Proof. The claim for $m = 1$ was proven in [2], Example 1. So let $m \ge 2$. Set $J_m = (q_0, \ldots, q_{2m})$. It suffices to show that $I(C_n) \subset \sqrt{J_m}$. In this proof, for all $f, g \in R$, by abuse of notation we will write $f \equiv g$ whenever $f - g$ or $f + g$ belongs to J_m, and, $f \equiv_{q_i} g$ whenever $f - g$ or $f + g$ is divisible by q_i. In this way, $f \equiv g$ or $f \equiv_{q_i} g$ assures that $f \in J_m$ occurs if and only if $g \in J_m$. We first show that

$$(2.1) \qquad\qquad x_1^{2^m} x_{3m+2}^{2^{m+1}} \in J_m.$$

Set

$$u_m \quad = \quad x_1^{2^{m-1}} x_{3m+2}^{2^m},$$
$$v_m \quad = \quad x_3 x_4 x_5 \prod_{i=2}^{m} x_{3i}^{3 \cdot 2^{i-2}},$$
$$w_m \quad = \quad (x_{3m}x_{3m+1}x_{3m+2})^{2^{m-1}}.$$

We prove that

$$(2.2) \qquad\qquad u_m \equiv v_m \equiv w_m.$$

Note that $x_1 x_{3m+2} v_m$ is a multiple of $x_1 x_{3m+2} x_4 x_5$, and

$$x_1 x_{3m+2} x_4 x_5 \equiv_{q_0} x_1 x_{3m+2}(x_2 x_3 + x_4 x_5) \in J_m,$$

whence we deduce that $x_1 x_{3m+2} v_m \in J_m$. Thus (2.2) will imply that

$$x_1^{2^m} x_{3m+2}^{2^{m+1}} = x_1 x_{3m+2} u_m \in J_m,$$

as claimed in (2.1). We prove (2.2) by induction on $m \geq 2$. First take $m = 2$. We have $q_2 = x_3 x_4 + x_5 x_6$, $q_3 = x_5 x_6 + x_7 x_8$ and $q_4 = x_1 x_8 + x_6 x_7$, so that

$$v_2 = x_3 x_4 x_5 x_6^3 \equiv_{q_2} x_5^2 x_6^2 x_6^2 \equiv_{q_3} x_6^2 x_7^2 x_8^2 = w_2 \equiv_{q_4} x_1^2 x_8^4 = u_2,$$

which shows (2.2) for $m = 2$. Now suppose that $m > 2$ and that the claim is true for $m - 1$. We have:

$$
\begin{aligned}
v_m \quad &= \quad v_{m-1} x_{3m}^{3 \cdot 2^{m-2}} \equiv w_{m-1} x_{3m}^{3 \cdot 2^{m-2}} \\[2mm]
&= \quad (x_{3m-3} x_{3m-2} x_{3m-1})^{2^{m-2}} x_{3m}^{3 \cdot 2^{m-2}} = (x_{3m-3} x_{3m-2})^{2^{m-2}} x_{3m-1}^{2^{m-2}} x_{3m}^{3 \cdot 2^{m-2}} \\[2mm]
&\equiv_{q_{2m-2}} \quad (x_{3m-1} x_{3m})^{2^{m-2}} x_{3m-1}^{2^{m-2}} x_{3m}^{3 \cdot 2^{m-2}} = (x_{3m-1} x_{3m})^{2^{m-1}} x_{3m}^{2 \cdot 2^{m-2}} \\[2mm]
&\equiv_{q_{2m-1}} \quad (x_{3m+1} x_{3m+2})^{2^{m-1}} x_{3m}^{2^{m-1}} = (x_{3m} x_{3m+1})^{2^{m-1}} x_{3m+2}^{2^{m-1}} = w_m \\[2mm]
&\equiv_{q_{2m}} \quad (x_1 x_{3m+2})^{2^{m-1}} x_{3m+2}^{2^{m-1}} = x_1^{2^{m-1}} x_{3m+2}^{2^m} = u_m.
\end{aligned}
$$

This completes the proof of (2.2) and of (2.1). We have thus shown that

$$(2.3) \qquad\qquad x_1 x_{3m+2} \in \sqrt{J_m}.$$

But then

$$(2.4) \qquad\qquad x_{3m} x_{3m+1} = q_{2m} - x_1 x_{3m+2} \in \sqrt{J_m}.$$

In general, whenever, for some $i \in \{2, \ldots, m\}$,

$$(2.5) \qquad\qquad x_{3i} x_{3i+1} \in \sqrt{J_m},$$

from the fact that $x_{3i} x_{3i+1}$ divides $x_{3i-1} x_{3i} \cdot x_{3i+1} x_{3i+2}$, i.e., the product of the summands of q_{2i-1}, by Lemma 2.1 one deduces that

$$(2.6) \qquad\qquad x_{3i-1} x_{3i} \in \sqrt{J_m}.$$

Since $x_{3i-3} x_{3i-2} = q_{2i-2} - x_{3i-1} x_{3i}$, this in turn implies that

$$(2.7) \qquad\qquad x_{3i-3} x_{3i-2} \in \sqrt{J_m}.$$

Finally, since $x_{3i-3} x_{3i-2}$ divides $x_{3i-4} x_{3i-3} \cdot x_{3i-2} x_{3i-1}$, i.e., the product of the summands of q_{2i-3}, by Lemma 2.1 we again conclude that

$$(2.8) \qquad\qquad x_{3i-2} x_{3i-1} \in \sqrt{J_m}.$$

Therefore, since (2.5) implies (2.6), (2.7) and (2.8), for all $i = 2, \ldots, m$, from (2.4) one can derive by descending induction on h, that $x_h x_{h+1} \in \sqrt{J_m}$ for all $h = 3, \ldots, 3m + 1$. In particular we have that $x_3 x_4 \in \sqrt{J_m}$, which, together with $q_1 \in J_m$, yields $x_2 x_3 \in \sqrt{J_m}$ by Lemma 2.1. This, together with (2.4) and $q_0 \in J_m$, shows that $I(C_n) \subset \sqrt{J_m}$, as claimed. $\qquad\square$

Theorem B and Propositions 2.2, 2.3, 2.4 imply our main result.

Theorem 2.5. *Let $n \geq 3$ be an integer. Then*

$$
ara\ I(C_n) = pd\ I(C_n) = \begin{cases} \frac{2n}{3} & if \quad n \equiv 0 \bmod 3 \\[2mm] \frac{2n+1}{3} & if \quad n \equiv 1 \bmod 3 \\[2mm] \frac{2n-1}{3} & if \quad n \equiv 2 \bmod 3. \end{cases}
$$

Every ideal $I(C_n)$ is of pure height $\lceil \frac{n}{2} \rceil$, where $\lceil a \rceil$ denotes the least integer not less than a. Recall that an ideal is called a *set-theoretic complete intersection* if its arithmetical rank equals its height. In view of Theorem 2.5 we thus have the following.

Corollary 2.6. $I(C_n)$ *is a set-theoretic complete intersection only for $n = 3$ and $n = 5$.*

3. The arithmetical rank of bicyclic graphs

In this section by $\equiv$, we mean $\equiv$ (mod 3) and all equivalence relations will be considered modulo 3. Let $a_1, \ldots, a_s$ be subsets of the finite set V. Define $\varepsilon(a_1, \ldots, a_s; V)$ to be the simplicial complex which has vertex set $\bigcup_{i=1}^{s} (V \setminus a_i)$ and maximal faces $V \setminus a_1, \ldots, V \setminus a_s$. Let $\Delta = \Delta_G$, and let $F \in \Delta^*$ and $e_1, \ldots, e_r$ be all the edges of G which are disjoint from F. Then $Link_{\Delta^*} F = \varepsilon(e_1, \ldots, e_r; V(G) \setminus F)$ by [12, Proposition 3.3]. According to [11, Proposition 6.1.6], associating F with the induced subgraph H of G on the vertex set $V(G) \setminus F$ defines a bijection between the faces of Δ^* and the set of induced subgraphs of G which have at least one edge. Let H be an induced subgraph of the graph G. If H is associated with the face F of Δ^* as described above, we write $\varepsilon(H)$ for $\varepsilon(e_1, \ldots, e_s; V)$, where $e_1, \ldots, e_s$ are the edges of H and V is the vertex set $V(G) \setminus F$ (or equivalently the vertex set of H). In this section, using (*), we find explicit descriptions of the projective dimension of all bicyclic graphs. For every vertex u of a graph G we denote by $N_G(u)$ the set of vertices adjacent to u. In the proof of our main results we will use the Mayer-Vietoris sequence for the reduced homology of simplicial complexes, which, for any pair Δ_1, Δ_2 of simplicial complexes, has the following form (see [11, Remark 6.2.13]):

$$
\ldots \to \tilde{H}_i(\Delta_1 \cap \Delta_2; K) \to \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \to \tilde{H}_i(\Delta_1 \cup \Delta_2; K) \to
$$
$$
\tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \to \ldots.
$$

Lemma 3.1. *For a graph G with an edge $\{u, v\}$ such that $deg(v) = 1$, we have $\tilde{H}_i(\varepsilon(G); K) = 0$, if some vertex in $N_G(u)$ has an adjacent vertex of degree one in G. Otherwise, $\tilde{H}_i(\varepsilon(G); K) = \tilde{H}_{i-t}(\varepsilon(H); K)$, where $t = |N_G(u)|$ and H is the induced subgraph on $V(G) \setminus (\{u\} \cup N_G(u))$, provided H is non empty.*

Proof. In this and in the following proofs we will omit the coefficient field in the homology groups. We set $V = V(G)$. Let $N_G(u) = \{v, u_1, \ldots, u_{t-1}\}$ and $\{u, v\}, \{u, u_1\}, \ldots, \{u, u_{t-1}\}, e_1, \ldots, e_r$ be the edges of G. We can write $\varepsilon(G) = E_1 \cup E_2$, where $E_1 = \varepsilon(\{u, u_1\}, \ldots, \{u, u_{t-1}\}, e_1, \ldots, e_r; V)$ and $E_2 = \varepsilon(\{u, v\}; V)$. The intersection of these simplicial complexes is:
$E_1 \cap E_2 = \varepsilon(\{u, v, u_1\}, \ldots, \{u, v, u_{t-1}\}, \{u, v\} \cup e_1, \ldots, \{u, v\} \cup e_r; V)$
$= \varepsilon(\{u_1\}, \ldots, \{u_{t-1}\}, e_1, \ldots, e_r; V \setminus (\{u, v\}))$ (see [12, Lemma 3.4]).
If there exists a vertex v_i of degree one such that $\{u_i, v_i\} \in E(G)$, then without loss

of generality we can assume that $e_1 = \{u_i, v_i\}$. Then $E_1 \cap E_2 = \varepsilon(\{u_1\}, \ldots, \{u_{t-1}\},$ $\{u_i, v_i\}, e_2, \ldots, e_r; V \setminus (\{u, v\})) = \varepsilon(\{u_1\}, \ldots, \{u_{t-1}\}, e_2, \ldots, e_r; V \setminus (\{u, v\}))$, whose reduced homology is identically zero, since $v_i \in V \setminus (\{u, v\})$ and v_i belongs to all faces of $E_1 \cap E_2$. Otherwise, by [12, Lemma 3.5] we have $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-t+1}(\varepsilon(H))$, for all i, where H is the induced subgraph on $V \setminus (\{u\} \cup N_G(u))$. Since E_2 is a simplex, $\tilde{H}_i(E_2) = 0$ for all i. Also, $\tilde{H}_i(E_1) = 0$ for all i, since v belongs to all faces of E_1. Using the Mayer-Vietoris sequence (for $\Delta_i = E_i$) we deduce that $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_{i-1}(E_1 \cap E_2)$, which completes the proof. □

The next result can be deduced from Lemma 3.1 by a trivial inductive argument.

Corollary 3.2. *Let $n \equiv 0$. Suppose that L_n intersects graph G only at one of its endpoints. Then, for all i, we have $\tilde{H}_i(\varepsilon(G \cup L_n)) = \tilde{H}_{i - \frac{2n}{3}}(\varepsilon(G \setminus L_n))$.*

Theorem 3.3. *Let G be the graph which is a joint of two cycles C_n and C_m in a common vertex. Then the following hold:*

(a) *If $|V(G)| \equiv 1$, then pd $I(G) = $ ara $I(G) = \frac{2|V(G)|+1}{3}$.*
(b) *If $|V(G)| \equiv 0$, then pd $I(G) = $ ara $I(G) = \frac{2|V(G)|}{3}$.*
(c) *If $|V(G)| \equiv 2$, then pd $I(G) = $ ara $I(G) = \frac{2|V(G)|+2}{3}$, for $m \equiv 0$, whereas pd $I(G) = $ ara $I(G) = \frac{2|V(G)|-1}{3}$ otherwise.*

Proof. We will prove the claim by showing that the desired number is, on the one hand, a lower bound for $pd\ I(G)$, on the other hand, an upper bound for $ara\ I(G)$.

Let $V = V(G)$. Consider the labeling for V such that $V(C_n) = \{y_1, y_2, \ldots, y_n\}$, and $V(C_m) = \{x_1, x_2, \ldots, x_m\}$, where $x_1 = y_1$. Up to exchanging m and n we have the following cases.

Case 1. $|V| \equiv 0$ or 1.
First let $n = 3$. Then $m \equiv 1$ or $m \equiv 2$. In view of (*) the ith Betti number of degree $|V|$ is $\beta_{i,|V|}(G) = dim_k \tilde{H}_{i-2}(\varepsilon(G); K)$. So we compute the reduced homology of G of degree $|V|$. We can write $\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{x_1, y_2\}, \{y_2, y_3\}, \{y_3, x_1\};$ $V) = E_1 \cup E_2$, where $E_1 = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{x_1, y_2\}, \{y_3, x_1\}; V)$ and $E_2 = \varepsilon(\{y_2, y_3\}; V)$.
By [12, Lemma 3.4], the intersection of these simplicial complexes is:

$$E_1 \cap E_2 = \varepsilon(\{x_1, x_2, y_2, y_3\}, \ldots, \{x_1, x_m, y_2, y_3\}, \{x_1, y_2, y_3\}; V)$$
$$= \varepsilon(\{x_1\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}; V \setminus \{y_2, y_3\}).$$

By [12, Lemma 3.5], $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-1}(\varepsilon(L_{m-1}))$, for all i. Since E_2 is a simplex, $\tilde{H}_i(E_2) = 0$ for all i. Applying Lemma 3.1 to E_1 for $v = y_2$ (and $u = x_1$, so that $N(u) = \{x_2, x_m, y_2, y_3\}$), we obtain $\tilde{H}_i(E_1) = \tilde{H}_{i-4}(\varepsilon(L_{m-3}))$, for all i.
If $m \equiv 1$, then $\tilde{H}_i(E_1) = 0$ for all i by Theorem D and (*) (since $m - 3 \equiv 1$). By the Mayer-Vietoris sequence we deduce that, for all i, $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_{i-1}(E_1 \cap E_2)$. Theorem D and (*) then show that $\tilde{H}_{i-2}(\varepsilon(L_{m-1})) \neq 0$ (i.e., $\tilde{H}_i(\varepsilon(G)) \neq 0$) if and only if $i - 2 + 2 = \frac{2(m-1)}{3}$, if and only if $i = \frac{2|V|}{3} - 2$. In view of (*) we deduce that $pd\ I(G) \geq \frac{2|V|}{3}$.
If $m \equiv 2$, then $\tilde{H}_i(E_1 \cap E_2) = 0$ for all i by Theorem D and (*) (since $m-1 \equiv 1$). By the Mayer-Vietoris sequence we deduce that $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_i(E_1)$ for all i. Theorem D and (*) show that $\tilde{H}_i(\varepsilon(G)) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 2$. In view of (*) we

deduce that $pd\, I(G) \geq \frac{2|V|+1}{3}$.

So we can assume that $n \geq 4$. Moreover, since n and m cannot be both divisible by 3, we may assume that $m \equiv 1$ or $m \equiv 2$. In view of (*) we compute the reduced homology of G of degree $|V|$. We can write

$\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \dots, \{x_m, x_1\}, \{x_1, y_2\}, \{y_2, y_3\}, \dots, \{y_{n-1}, y_n\}, \{y_n, x_1\}; V) = E_1 \cup E_2$, where $E_1 = \varepsilon(\{x_2, x_3\}, \dots, \{x_m, x_1\}, \{x_1, y_2\}, \dots, \{y_{n-1}, y_n\}, \{y_n, x_1\}; V)$ and $E_2 = \varepsilon(\{x_1, x_2\}; V)$. We have that $E_1 = \varepsilon(L_m \cup C_n)$, where $L_m : x_2 x_3 \dots x_m x_1$. The intersection of these simplicial complexes is $E_1 \cap E_2 = \varepsilon(\{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \dots, \{x_1, x_2, x_{m-1}, x_m\}, \{x_m, x_1, x_2\}, \{x_1, y_2, x_2\}, \dots, \{y_{n-1}, y_n, x_1, x_2\}, \{y_n, x_1, x_2\}; V) = \varepsilon(\{x_3\}, \{x_m\}, \{y_2\}, \{y_n\}, \{x_4, x_5\}, \dots, \{x_{m-2}, x_{m-1}\}, \{y_3, y_4\}, \dots, \{y_{n-2}, y_{n-1}\}; V \setminus \{x_1, x_2\})$ (see [12, Lemma 3.4]).

By [12, Lemma 3.5], $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-4}(\varepsilon(H))$ for all i, where H is the induced subgraph on $V \setminus \{x_1, x_2, x_3, x_m, y_2, y_n\}$. We have $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-4}(\varepsilon(L_{m-4} \cup L_{n-3}))$ for all i. Since E_2 is a simplex, $\tilde{H}_i(E_2) = 0$ for all i.

Case 1.1. Let $m \equiv 2$.

By Theorem C, $\tilde{H}_i(E_1 \cap E_2) = 0$ for any i, since $m - 4 \equiv 1$. Applying Corollary 3.2 to the path $L_{m-2} : x_2 x_3 \dots x_{m-1}$, we get that, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m-2)}{3}}(\varepsilon(L_2 \cup C_n))$, where $L_2 : x_1 x_m$. If we apply Lemma 3.1 once again for $v = x_m$ (and $u = x_1$, so that $N(u) = \{x_m, y_2, y_n\}$), we then obtain $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m-2)}{3}-3}(\varepsilon(L_{n-3}))$, for all i, where $L_{n-3} : y_3 \dots y_{n-1}$. In part (a), we have $n \equiv 0$. Theorem D and (*) show that $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 2$. The Mayer-Vietoris sequence implies that $\tilde{H}_i(\varepsilon(G)) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 2$. By (*) it follows that $pd\, I(G) \geq \frac{2|V|+1}{3}$, as claimed. In part (b), we have $n \equiv 2$. Theorem D and (*) show that $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|}{3} - 2$. As above, it follows that $\tilde{H}_i(\varepsilon(G)) \neq 0$ if and only if $i = \frac{2|V|}{3} - 2$. In view of (*) we deduce that $pd\, I(G) \geq \frac{2|V|}{3}$, as claimed.

Case 1.2. Let $m \equiv 1$.

By Theorem C, $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-4-\frac{2(m-4)}{3}}(\varepsilon(L_{n-3}))$ for any i. Moreover, applying Corollary 3.2 to the path $L_{m-1} : x_2 x_3 \dots x_m$, we get that, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m-1)}{3}}(\varepsilon(C_n))$. In part (a), we have $n \equiv 1$. Hence, by Theorem C, $\tilde{H}_i(E_1 \cap E_2) = 0$ for any i, since $n - 3 \equiv 1$. On the other hand, Theorem B and (*) show that $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 2$. We deduce that $\tilde{H}_i(\varepsilon(G)) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 2$. By (*) it follows that $pd\, I(G) \geq \frac{2|V|+1}{3}$, as claimed. In part (b), we have $n \equiv 0$. By Theorem D and (*) it follows that $\tilde{H}_i(E_1 \cap E_2) \neq 0$ if and only if $i = \frac{2|V|}{3} - 2$, in which case the ith homology module is equal to K. Theorem B and (*) show that $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|}{3} - 2$ in which case it is equal to K^2. Thus the Mayer-Vietoris sequence implies that $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|}{3} - 2$. In view of (*) we deduce that $pd\, I(G) \geq \frac{2|V|}{3}$, as claimed.

Case 2. Let $|V| \equiv 2$.

Case 2.1. $m \equiv 0$.

We have $n \equiv 0$. First assume that $n = m = 3$. We first show that in this case $pd\, I(G) \geq 4$. We use the fact that $pd\, I(G) = cd\, I(G)$, (see [15, Theorem 1]), where cd denotes the local cohomological dimension, i.e., for any ideal I of

R, $cd\ I$ is the maximum index i for which the local cohomology module $H_I^i(R)$ (of R with respect to I) does not vanish. We have that $I(G) = I \cap J$, where $I = (x_1, x_2x_3, y_2y_3)$ and $J = (x_2, x_3, y_2, y_3)$. It is well-known that, whenever an ideal is a complete intersection, its height is the only index for which the cohomology module of R with respect to this ideal does not vanish (see [10, Proposition 3.8] together with [10, Theorem 4.4], or together with [8, Example 2, p. 414]). Since $I + J = (x_1, x_2, x_3, y_2, y_3)$ we thus have that $H_{I+J}^i(R) \neq 0$ if and only if $i = 5$. We also have that $cd\ J = 4$. In the Mayer-Vietoris sequence for local cohomology (see [10], Section 3)

$$\ldots \to H_{I+J}^4(R) \to H_I^4(R) \oplus H_J^4(R) \to H_{I\cap J}^4(R) \to \ldots,$$

the left term is zero, whereas the middle term is not. It follows that the right term is non zero, too. This implies that $pd\ I(G) = cd\ I(G) \geq 4$. On the other hand, by Lemma 2.1, the elements $x_1x_2, x_2x_3 + x_1x_3, x_1y_2, x_1y_3 + y_2y_3$ generate $I(G)$, up to radical, which shows that $ara\ I(G) \leq 4$. It follows that $pd\ I(G) = ara\ I(G) = 4$, which proves the claim in this case.

Without loss of generality we can thus assume that $m \geq 6$. We can write $\varepsilon(G) = E_1 \cup E_2$, where $E_1 = \varepsilon(\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_m, x_1\}, \{x_1, y_2\}, \ldots, \{y_n, x_1\}; V)$ and $E_2 = \varepsilon(\{x_2, x_3\}; V)$. We have that $E_1 \cap E_2 = \varepsilon(\{x_1\}, \{x_4\}, \{x_5, x_6\}, \ldots,$ $\{x_{m-1}, x_m\}, \{y_2, y_3\}, \ldots, \{y_{n-1}, y_n\})$. By [12, Lemma 3.5]) it follows that $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-2}(\varepsilon(L_{m-4} \cup L_{n-1}))$, for all i. We also have that $E_1 = \varepsilon(H_1)$, where H_1 is the union of C_n and the paths $x_3 \ldots x_m x_1$ and $x_1 x_2$. Applying Lemma 3.1 for $v = x_2$ (and $u = x_1$, so that $N(u) = \{x_2, x_m, y_2, y_n\}$, we obtain that, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i-4}(\varepsilon(L_{m-3} \cup L_{n-3}))$. Thus, by Theorem C, we deduce that, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i-4-\frac{2(n-3)}{3}}(\varepsilon(L_{m-3}))$. According to Theorem D and (*) it is non zero if and only if $i = \frac{2|V|+2}{3} - 2$. Applying Theorem C, Theorem D and (*) we also get that $\tilde{H}_i(E_1 \cap E_2) \neq 0$ if and only if $i = \frac{2|V|+2}{3} - 4$. By the Mayer-Vietoris sequence we conclude that $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|+2}{3} - 2$. In view of (*) we deduce that $pd\ I(G) \geq \frac{2|V|+2}{3}$, as claimed.

Case 2.2. Let $m \equiv 1$.
We have $n \equiv 2$. Consider the induced subgraph H' on $V \setminus \{y_2\}$. Then H' is the union of C_m and the path $L_{n-1} : y_3 \ldots y_n y_1$. Applying Corollary 3.2 to the path $L_{n-2} : y_3 \ldots y_n$, we obtain $\tilde{H}_i(\varepsilon(H')) = \tilde{H}_{i-\frac{2(n-2)}{3}}(\varepsilon(C_m))$, for all i. By Theorem B and (*) it is non zero if and only if $i = \frac{2|V|-1}{3} - 2$. In view of (*) we deduce that $pd\ I(G) \geq \frac{2|V|-1}{3}$, as claimed.

Now we find an upper bound for the arithmetical rank in each case. In the rest of the proof, we will refer to the polynomials q_i introduced in Propositions 2.2, 2.3 and 2.4; in each case, the polynomial q_i' will be the one obtained from q_i by replacing each variable x_j with y_j.

In part (a), for $m \equiv 2$, by Proposition 2.4 the sequence $A_m : q_0, \ldots, q_{\frac{2(m-2)}{3}}$, generates $I(C_m)$, up to radical and by Proposition 2.2 the sequence $A_n : q_0', \ldots, q_{\frac{2n}{3}-1}'$, generates $I(C_n)$, up to radical. Since $I(G) = I(C_m) + I(C_n)$, the following sequence generates $I(G)$, up to radical: $B : q_0, \ldots, q_{\frac{2(m-2)}{3}}, q_0', \ldots, q_{\frac{2n}{3}-1}'$. This implies that

$ara\ I(G) \le \frac{2|V|+1}{3}$.

If $m \equiv 1$, then, by Proposition 2.3 the sequence $A_m :\ q_0, \ldots, q_{\frac{2(m-1)}{3}}$ generates $I(C_m)$, up to radical and the sequence $A_n :\ q'_0, \ldots, q'_{\frac{2(n-1)}{3}}$ generates $I(C_n)$, up to radical. The summand $x_1 x_m$ of q_1 divides the product of the monomials $q_{\frac{2(m-1)}{3}} = x_{m-1} x_m$ and $q'_0 = y_1 y_2 = x_1 y_2$. Thus by Lemma 2.1 the sequence $B :\ q_0, q_1, q_{\frac{2(m-1)}{3}} + q'_0, q_2, \ldots, q_{\frac{2(m-1)}{3}-1}, q'_1, \ldots, q'_{\frac{2(n-1)}{3}}$ of length $\frac{2|V|+1}{3}$ generates $I(G)$ up to radical. This implies that $ara\ I(G) \le \frac{2|V|+1}{3}$.

In part (b), for $m \equiv 2$, according to Proposition 2.4, the sequence $A_m :\ q_0, \ldots, q_{\frac{2(m-2)}{3}}$, generates $I(C_m)$, up to radical and the sequence $A_n :\ q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$, generates $I(C_n)$, up to radical. The sequence B formed by the union of these two sequences generates $I(G)$, up to radical. This implies that $ara\ I(G) \le \frac{2|V|}{3}$.

If $m \equiv 1$, then, by Proposition 2.3 the sequence $A_m :\ q_0, \ldots, q_{\frac{2(m-1)}{3}}$ generates $I(C_m)$, up to radical and by Proposition 2.2 the sequence $A_n :\ q'_0, \ldots, q'_{\frac{2n}{3}-1}$ generates $I(C_n)$, up to radical. Thus by Lemma 2.1 the sequence $B :\ q_0, q_1, q_{\frac{2(m-1)}{3}} + q'_0, q_2, \ldots, q_{\frac{2(m-1)}{3}-1}, q'_1, \ldots, q'_{\frac{2n}{3}-1}$ of length $\frac{2|V|}{3}$, generates $I(G)$, up to radical. So $ara\ I(G) \le \frac{2|V|}{3}$.

In part (c), if $m \equiv 0$, then consider the sequence $B :\ q_0, \ldots, q_{\frac{2m}{3}-1}, q'_0, \ldots, q'_{\frac{2n}{3}-1}$, where $A_m :\ q_0, \ldots, q_{\frac{2m}{3}-1}$ generates $I(C_m)$ and $A_n :\ q'_0, \ldots, q'_{\frac{2n}{3}-1}$ generates $I(C_n)$, up to radical, by Proposition 2.2. This implies that $ara\ I(G) \le \frac{2|V|+2}{3}$.

If $m \equiv 1$, then, by Proposition 2.3, the sequence $A_m :\ q_0, \ldots, q_{\frac{2(m-1)}{3}}$, generates $I(C_m)$, up to radical. By Proposition 2.4, the sequence $A_n :\ q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$, generates $I(C_n)$, up to radical. Thus by Lemma 2.1 the sequence $B :\ q_0, q_1, q_{\frac{2(m-1)}{3}} + q'_0, q_2, \ldots, q_{\frac{2(m-1)}{3}-1}, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}$, generates $I(G)$, up to radical. This implies that $ara\ I(G) \le \frac{2|V|-1}{3}$. $\square$

Theorem 3.4. *Let G be the graph formed by two cycles C_n and C_m with a path joining a vertex of C_n to a vertex of C_m. Then the following hold:*

(a) If $|V(G)| \equiv 1$, then pd $I(G) = ara\ I(G) = \frac{2|V(G)|-2}{3}$, whenever $m \equiv 2$, $n \equiv 2$. Otherwise, pd $I(G) = ara\ I(G) = \frac{2|V(G)|+1}{3}$.

(b) If $|V(G)| \equiv 0$, then pd $I(G) = ara\ I(G) = \frac{2|V(G)|}{3}$.

(c) If $|V(G)| \equiv 2$ and $m, n \equiv 0$ or 1, then pd $I(G) = ara\ I(G) = \frac{2|V(G)|+2}{3}$. Otherwise, pd $I(G) = ara\ I(G) = \frac{2|V(G)|-1}{3}$.

Proof. Let $V = V(G)$. Consider the labeling for V such that $V(C_n) = \{y_1, y_2, \ldots, y_n\}$, $V(C_m) = \{x_1, x_2, \ldots, x_m\}$ and let $P :\ z_0 z_1 \ldots z_k z_{k+1}$ be the path in G, where $z_0 = x_1$ and $z_{k+1} = y_1$. We compute the reduced homology of G of degree $|V|$. Up to exchanging m and n, we have the following cases.

Case 1. Let $k \equiv 2$.

We can write $\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{y_1, y_2\}, \ldots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\},$
$\ldots, \{z_{k-1}, z_k\}, \{z_k, y_1\}; V) = E_1 \cup E_2$, where
$E_1 = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{y_1, y_2\}, \ldots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-1}, z_k\};$
$V)$ and $E_2 = \varepsilon(\{z_k, y_1\}; V)$. The intersection of these simplicial complexes is
$E_1 \cap E_2 = \varepsilon(\{y_2\}, \{y_n\}, \{z_{k-1}\}, \{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{y_3, y_4\}, \ldots, \{y_{n-2}, y_{n-1}\},$
$\{x_1, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-3}, z_{k-2}\}; V)$ (see [12, Lemma 3.4]).
By [12, Lemma 3.5] it follows that

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-3}(\varepsilon(H_1 \cup L_{n-3})),$$

for all i, where H_1 is the induced subgraph on $V \setminus (V(C_n) \cup \{z_{k-1}, z_k\})$. Applying
Corollary 3.2 to the path $L_{k-2}: z_1 \ldots z_{k-2}$, we have

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i - \frac{2(k-2)}{3} - 3}(\varepsilon(C_m \cup L_{n-3})),$$

for all i. Since E_2 is a simplex, $\tilde{H}_i(E_2) = 0$ for all i.
Applying Corollary 3.2 to the path $L_{k+1}: z_0 \ldots z_k$, we have that, for all i,

$$\tilde{H}_i(E_1) = \tilde{H}_{i - \frac{2(k+1)}{3}}(\varepsilon(L_{m-1} \cup C_n)).$$

Case 1.1 Let $n \equiv 1$.

By Theorem C, since $n - 3 \equiv 1$, we have that

$$\tilde{H}_i(E_1 \cap E_2) = 0$$

for all i. The Mayer-Vietoris sequence then implies that $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_i(E_1)$ for all
i. Moreover, in view of Proposition 2.3, $I(C_n)$ is generated, up to radical, by the
sequence $A_n: q'_0, \ldots, q'_{\frac{2(n-1)}{3}}$.

Case 1.1.1 Let $m \equiv 1$ or $m \equiv 0$.

First suppose that $m \equiv 1$. Then $|V| \equiv 1$. From Theorem C we have that, for all i,
$\tilde{H}_i(E_1) = \tilde{H}_{i - \frac{2(k+1)}{3} - \frac{2(m-1)}{3}}(\varepsilon(C_n))$. In view of Theorem B and (*) it follows that
$\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 2$. Thus $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|+1}{3} - 2$,
which, by (*), implies that $pd\ I(G) \geq \frac{2|V|+1}{3}$.
By Lemma 2.1, the sequence $B: q_0, q_1, q_{\frac{2(m-1)}{3}} + x_1 z_1, q_2, \ldots, q_{\frac{2(m-1)}{3} - 1}, z_k y_1, z_{k-1} z_k$
$+ q'_0, q'_1, \ldots, q'_{\frac{2(n-1)}{3}}, z_2 z_3, z_1 z_2 + z_3 z_4, \ldots, z_{k-3} z_{k-2}, z_{k-4} z_{k-3} + z_{k-2} z_{k-1}$ of length
$\frac{2|V|+1}{3}$, generates $I(G)$, up to radical, where $A_m: q_0, \ldots, q_{\frac{2(m-1)}{3}}$ generates $I(C_m)$,
up to radical by Proposition 2.3. Thus $ara\ I(G) \leq \frac{2|V|+1}{3}$.
Now suppose that $m \equiv 0$. Then $|V| \equiv 0$. From Theorem C, Theorem B and (*)
we deduce that $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|}{3} - 2$. Thus $\tilde{H}_i(\varepsilon(G)) \neq 0$ for
$i = \frac{2|V|}{3} - 2$, which, by (*), implies that $pd\ I(G) \geq \frac{2|V|}{3}$.
By Lemma 2.1, the sequence $B: q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + y_1 z_k, q'_2, \ldots, q'_{\frac{2(n-1)}{3} - 1}, z_1 x_1, z_1 z_2 +$
$q_0, q_1, \ldots, q_{\frac{2m}{3} - 1}, z_3 z_4, z_2 z_3 + z_4 z_5, \ldots, z_{k-2} z_{k-1}, z_{k-3} z_{k-2} + z_{k-1} z_k$ of length $\frac{2|V|}{3}$,
generates $I(G)$, up to radical, where $A_m: q_0, \ldots, q_{\frac{2m}{3}}$ generates $I(C_m)$, up to
radical by Proposition 2.2. Therefore, we have $pd\ I(G) = ara\ I(G) = \frac{2|V|}{3}$.

Case 1.1.2 Let $m \equiv 2$.

In this case $|V| \equiv 2$. Consider the induced subgraph H_2 on $V \setminus \{z_k\}$. We have

$H_2 = H' \cup C_n$, where H' is the induced subgraph on $V \setminus (V(C_n) \cup \{z_k\})$. By Theorem A we have that, for all i, $\tilde{H}_i(\varepsilon(H_2)) = \tilde{H}_{i - \frac{2n+1}{3}}(\varepsilon(H'))$. If we apply Corollary 3.2 along the path $L_{k-2} : z_2 \ldots z_{k-1}$, and then Lemma 3.1 for $v = z_1$, we further get, for all i, $\tilde{H}_i(\varepsilon(C_n \cup H')) = \tilde{H}_{i - \frac{2n+1}{3} - \frac{2(k-2)}{3} - 3}(\varepsilon(L_{m-3}))$, which, by Theorem D and (*), is non zero in $i = \frac{2|V|-1}{3} - 2$. By (*) this implies $pd\ I(G) \geq \frac{2|V|-1}{3}$.

By Lemma 2.1, the sequence $B :\ z_1 x_1, z_1 z_2 + q_0, q_1, \ldots, q_{\frac{2(m-2)}{3}}, z_3 z_4, z_2 z_3 + z_4 z_5, \ldots, z_{k-2} z_{k-1}, z_{k-3} z_{k-2} +$

$z_{k-1} z_k, q_0', q_1', z_k y_1 + q_{\frac{2(n-1)}{3}}', q_2', \ldots, q_{\frac{2(n-1)}{3} - 1}'$ of length $\frac{2|V|-1}{3}$ generates $I(G)$, up to radical, where the sequence $A_n :\ q_0', \ldots, q_{\frac{2(n-1)}{3}}'$ generates $I(C_n)$, up to radical, by Proposition 2.3. This shows that $ara\ I(G) \leq \frac{2|V|-1}{3}$.

Case 1.2 Let $n \equiv 2$.

By Theorem C, since $n - 3 \equiv 2$, we have that

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i - \frac{2(k-2)}{3} - 3 - \frac{2(n-3)-1}{3}}(\varepsilon(C_m))$$

for all i. Moreover, by Theorem A,

$$\tilde{H}_i(E_1) = \tilde{H}_{i - \frac{2(k+1)}{3} - \frac{2n-1}{3}}(\varepsilon(L_{m-1})),$$

for all i. Moreover, in view of Proposition 2.4, $I(C_n)$ is generated, up to radical, by the sequence $A_n :\ q_0', \ldots, q_{\frac{2(n-2)}{3}}'$.

Case 1.2.1 Let $m \equiv 0$.

In this case $|V| \equiv 1$. In view of Theorem B and (*) we have that $\tilde{H}_i(E_1 \cap E_2) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 3$, in which case the homology group is K^2, and, according to Theorem D and (*), we have that $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 3$, in which case the homology group is K. From the Mayer-Vietoris sequence it then follows that $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|+1}{3} - 2$, which, by (*), implies that $pd\ I(G) \geq \frac{2|V|+1}{3}$. The sequence $B : q_0', \ldots, q_{\frac{2(n-2)}{3}}', q_0, \ldots, q_{\frac{2m}{3} - 1}, z_1 z_2, z_0 z_1 + z_2 z_3, \ldots, z_{k-1} z_k, z_{k-2} z_{k-1}$

$+ z_k z_{k+1}$ of length $\frac{2|V|+1}{3}$ generates $I(G)$, up to radical, by Lemma 2.1, where the sequence $A_m :\ q_0, \ldots, q_{\frac{2m}{3} - 1}$ generates $I(C_m)$, up to radical, by Proposition 2.2. This implies that $ara\ I(G) \leq \frac{2|V|+1}{3}$.

Case 1.2.2 Let $m \equiv 2$.

In this case $|V| \equiv 0$. In view of Theorem B and (*) we have that $\tilde{H}_i(E_1 \cap E_2) \neq 0$ if and only if $i = \frac{2|V|}{3} - 3$, and, according to Theorem C, since $m - 1 \equiv 1$, we have that $\tilde{H}_i(E_1) = 0$ for all i. From the Mayer-Vietoris sequence it then follows that $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|}{3} - 2$, which, by (*), implies that $pd\ I(G) \geq \frac{2|V|}{3}$.

By Lemma 2.1, the sequence $B :\ q_0, \ldots, q_{\frac{2(m-2)}{3}}, q_0', \ldots, q_{\frac{2(n-2)}{3}}', z_1 z_2, z_0 z_1 + z_2 z_3, \ldots,$

$z_{k-1} z_k, z_{k-2} z_{k-1} + z_k z_{k+1}$ of length $\frac{2|V|}{3}$ generates $I(G)$, up to radical, where the sequence $A_m :\ q_0, \ldots, q_{\frac{2(m-2)}{3}}$ generates $I(C_m)$, up to radical, by Proposition 2.4. This implies that $ara\ I(G) \leq \frac{2|V|}{3}$.

Case 1.3 Let $n \equiv m \equiv 0$.

In this case $|V| \equiv 2$. In view of Theorem C, since $n - 3 \equiv 0$, we have that, for all i,

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i - \frac{2(k-2)}{3} - 3 - \frac{2(n-3)}{3}}(\varepsilon(C_m)),$$

and by Theorem A, for all i,

$$\tilde{H}_i(E_1) = \tilde{H}_{i - \frac{2(k+1)}{3} - \frac{2n}{3}}(\varepsilon(L_{m-1})).$$

According to Theorem B and (*) it follows that $\tilde{H}_i(E_1 \cap E_2) \neq 0$ if and only if $i = \frac{2|V|+2}{3} - 3$, in which case it is equal to K^2, and, in view of Theorem D and (*), $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|+2}{3} - 3$, in which case it is equal to K. From the Mayer-Vietoris sequence it then follows that $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|+2}{3} - 2$, which, by (*), implies that $pd\ I(G) \geq \frac{2|V|+2}{3}$.

The sequence $B: q'_0, \ldots, q'_{\frac{2n}{3} - 1}, q_0, \ldots, q_{\frac{2m}{3} - 1}, z_1 z_2, x_1 z_1 + z_2 z_3, \ldots, z_{k-1} z_k, z_{k-2} z_{k-1} + z_k y_1$, generates $I(G)$, up to radical, by Lemma 2.1, where $A_m: q_0, \ldots, q_{\frac{2m}{3} - 1}$ generates $I(C_m)$, up to radical, and $A_n: q'_0, \ldots, q'_{\frac{2n}{3} - 1}$ generates $I(C_n)$, up to radical, by Proposition 2.2. Therefore, we have $ara\ I(G) \leq \frac{2|V|+2}{3}$.

Case 2 Let $k \equiv 0$.
As in Case 1, we can write $\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{y_1, y_2\}, \ldots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-1}, z_k\}, \{z_k, y_1\}; V) = E_1 \cup E_2$, where $E_1 = \varepsilon(\{x_1, x_2\}, \ldots, \{x_1, x_m\}, \{y_1, y_2\}, \ldots, \{y_1, y_n\}, \{x_1, z_1\}, \ldots, \{z_{k-1}, z_k\})$ and $E_2 = \varepsilon(\{z_k, y_1\})$.
If $k = 0$, then $E_1 \cap E_2 = \varepsilon(\{x_2\}, \{x_m\}, \{y_2\}, \{y_n\}, \{x_3, x_4\}, \ldots, \{x_{m-2}, x_{m-1}\}, \{y_3, y_4\}, \ldots, \{y_{n-2}, y_{n-1}\}; V \setminus \{x_1, y_1\})$, so that, by [11, Lemma 3.5],

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-4}(\varepsilon(L_{m-3} \cup L_{n-3})),$$

for all i. If $k \geq 3$, then $E_1 \cap E_2 = \varepsilon(\{z_{k-1}\}, \{y_2\}, \{y_n\}, \{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{y_3, y_4\}, \ldots, \{y_{n-2}, y_{n-1}\}, \{x_1, z_1\}, \{z_1, z_2\}, \ldots \{z_{k-3}, z_{k-2}\}; V \setminus \{z_k, y_1\})$, so that, by [11, Lemma 3.5], $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-3}(\varepsilon(H'' \cup L_{n-3}))$, for all i, where H'' is the induced subgraph on $V \setminus (V(C_n) \cup \{z_{k-1}, z_k\})$, i.e., it is the union of C_m and the path $L_{k-1}: x_1 z_1 \ldots z_{k-2}$. If we apply Corollary 3.2 along the path $L_{k-3}: z_2 \ldots z_{k-2}$ and then Lemma 3.1 for $v = z_1$, we deduce that, for all i,

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-6-\frac{2(k-3)}{3}}(\varepsilon(L_{m-3} \cup L_{n-3})),$$

which is evidently also true for $k = 0$. If $k = 0$, we have that $E_1 = \varepsilon(C_m \cup C_n)$, otherwise, if we apply Corollary 3.2 along the path $L_k: z_1 \ldots z_k$, we obtain that, for all i,

$$\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2k}{3}}(C_m \cup C_n).$$

This equality is evidently also true for $k = 0$. Since E_2 is a simplex, we also have that $\tilde{H}_i(E_2) = 0$ for all i.

Case 2.1 Let $n \equiv 1$.
In view of Theorem D (for $m = 3$) and of Theorem C (for $m \geq 4$), since $n - 3 \equiv 1$, we have that $\tilde{H}_i(E_1 \cap E_2) = 0$ for all i, so that $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_i(E_1)$ for all i. Moreover, in view of Theorem A, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i - \frac{2k}{3} - \frac{2n+1}{3}}(\varepsilon(C_m))$.
By Proposition 2.3, the sequence $A_n: q'_0, q'_1, \ldots, q'_{\frac{2(n-1)}{3}}$ generates $I(C_n)$, up to radical.

Let $m \equiv 1$. Then $|V| \equiv 2$. In view of Theorem B and (*), $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|+2}{3} - 2$. Hence $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|+2}{3} - 2$, which, by (*), implies that $pd\ I(G) \geq \frac{2|V|+2}{3}$.

If $k = 0$, then the sequence $q_0, q_1, q_{\frac{2(m-1)}{3}} + x_1y_1, q_2, \ldots, q_{\frac{2(m-1)}{3}-1}, q'_0, \ldots, q'_{\frac{2(n-1)}{3}}$ of length $\frac{2|V|+2}{3}$, generates $I(G)$, up to radical, by Lemma 2.1, where the sequence $A_m:\ q_0, \ldots, q_{\frac{2(m-1)}{3}}$ generates $I(C_m)$, up to radical, by Proposition 2.3.

If $k \geq 3$, then the sequence $B:\ q_0, q_1, q_{\frac{2(m-1)}{3}} + x_1z_1, q_2, \ldots, q_{\frac{2(m-1)}{3}-1}, z_2z_3, z_1z_2 + z_3z_4, \ldots, z_{k-1}z_k, z_{k-2}z_{k-1} + z_ky_1, q'_0, \ldots, q'_{\frac{2(n-1)}{3}}$ of length $\frac{2|V|+2}{3}$, generates $I(G)$, up to radical. Hence we have $ara\ I(G) \leq \frac{2|V|+2}{3}$.

Let $m \equiv 2$. Then $|V| \equiv 0$. In view of Theorem B and (*), $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|}{3} - 2$. Hence $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|}{3} - 2$, which, by (*), implies that $pd\ I(G) \geq \frac{2|V|}{3}$.

If $k = 0$, then by Lemma 2.1, the sequence $B:\ q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + x_1y_1, q'_2, \ldots, q'_{\frac{2(n-1)}{3}-1}, q_0, \ldots, q_{\frac{2(m-2)}{3}}$, of length $\frac{2|V|}{3}$, generates $I(G)$ up to radical, where the sequence $A_m:\ q_0, \ldots, q_{\frac{2(m-2)}{3}}$ generates $I(C_m)$, up to radical, by Proposition 2.4.

If $k \geq 3$, then the sequence $B:\ q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + y_1z_k, q'_2, \ldots, q'_{\frac{2(n-1)}{3}-1}, z_1z_2, x_1z_1 + z_2z_3, \ldots, z_{k-2}z_{k-1}, z_{k-3}z_{k-2} + z_{k-1}z_k, q_0, \ldots, q_{\frac{2(m-2)}{3}}$, of length $\frac{2|V|}{3}$, generates $I(G)$ up to radical. This implies that $ara\ I(G) \leq \frac{2|V|}{3}$.

Let $m \equiv 0$. Then $|V| \equiv 1$. In view of Theorem B and (*), $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|+1}{3} - 2$. Hence $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|+1}{3} - 2$, which, by (*), implies that $pd\ I(G) \geq \frac{2|V|+1}{3}$.

If $k = 0$, then the sequence $B:\ q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + x_1y_1, q'_2, \ldots, q'_{\frac{2(n-1)}{3}-1}, q_0, \ldots, q_{\frac{2m}{3}-1}$ of length $\frac{2|V|+1}{3}$, generates $I(G)$, up to radical, by Lemma 2.1, where the sequence $A_m:\ q_0, \ldots, q_{\frac{2m}{3}-1}$ generates $I(C_m)$, up to radical, by Proposition 2.2.

If $k \geq 3$, then the sequence $B:\ q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + y_1z_k, q'_2, \ldots, q'_{\frac{2(n-1)}{3}-1}, z_1z_2, x_1z_1 + z_2z_3, \ldots, z_{k-2}z_{k-1}, z_{k-3}z_{k-2} + z_{k-1}z_k, q_0, \ldots, q_{\frac{2m}{3}-1}$ of length $\frac{2|V|+1}{3}$, generates $I(G)$, up to radical, by Lemma 2.1. This shows that $ara\ I(G) \leq \frac{2|V|+1}{3}$.

Case 2.2 Let $n \equiv 2$ and $m \equiv 0$.
In this case $|V| \equiv 2$. In view of Theorem C, Theorem D and (*), $\tilde{H}_i(E_1 \cap E_2) \neq 0$ if and only if $i = \frac{2|V|-1}{3} - 2$, in which case the homology group is K. Moreover, in view of Theorem A, $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2k}{3}-\frac{2m}{3}}(\varepsilon(C_n))$, for all i. In view of Theorem B and (*), $\tilde{H}_i(E_1) \neq 0$ if and only if $i = \frac{2|V|-1}{3} - 2$, in which case the homology group is K^2. Hence $\tilde{H}_i(\varepsilon(G)) \neq 0$ for $i = \frac{2|V|-1}{3} - 2$, which, by (*), implies that $pd\ I(G) \geq \frac{2|V|-1}{3}$.
Let $k = 0$. The sequence $B:\ x_1y_1, q_0 + q'_0, q_1, \ldots, q_{\frac{2m}{3}-1}, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}$ of length $\frac{2|V|-1}{3}$ generates $I(G)$, up to radical, by Lemma 2.1, where the sequence $A_m:\ q_0, q_1, \ldots, q_{\frac{2m}{3}-1}$ generates $I(C_m)$, up to radical, by Proposition 2.2.

Let $k \geq 3$. The sequence $B: x_1 z_1, q_0 + z_1 z_2, q_1, \ldots, q_{\frac{2m}{3}-1}, z_3 z_4, z_2 z_3 + z_4 z_5, \ldots,$
$z_{k-3} z_{k-2}, z_{k-4} z_{k-3} + z_{k-2} z_{k-1}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}$ of length $\frac{2|V|-1}{3}$
generates $I(G)$, up to radical. Hence, in view of (*), we conclude that $ara\, I(G) \leq \frac{2|V|-1}{3}$.

Case 2.3 Let $n \equiv m \equiv 2$.
In this case $|V| \equiv 1$. Consider the induced subgraph H_3 on $V \setminus \{x_2\}$. Applying
Corollary 3.2 to the path $L_{m+k-2}: x_3 x_4 \ldots x_1 z_1 \ldots z_{k-1}$ ($L_{m-2}: x_3 x_4 \ldots x_1$, if
$k = 0$) and Lemma 3.1 for $v = z_k$ ($u = y_1$), we obtain that, for all i, $\tilde{H}_i(\varepsilon(H_3)) = \tilde{H}_{i-\frac{2(m+k-2)}{3}-3}(\varepsilon(L_{n-3}))$, which, by Theorem D and (*), is non zero in $i = \frac{2|V|-2}{3}-2$.
So $\tilde{H}_{\frac{2|V|-2}{3}-2}(\varepsilon(G)) \neq 0$ and by (*) we have $pd\, I(G) \geq \frac{2|V|-2}{3}$.
If $k = 0$, then, by Lemma 2.1, the sequence $B: x_1 y_1, q_0 + q'_0, q_1, \ldots, q_{\frac{2(m-2)}{3}}, q'_1, \ldots,$
$q'_{\frac{2(n-2)}{3}}$ generates $I(G)$, up to radical, where the sequence $A_m: q_0, \ldots, q_{\frac{2(m-2)}{3}}$ gen-
erates $I(C_m)$, up to radical, and the sequence $A_n: q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$ generates $I(C_n)$,
up to radical, by Proposition 2.4.
If $k \geq 3$, then, by Lemma 2.1, the sequence $B: z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_0 z_1,$
$z_1 z_2 + q_0, q_1, \ldots, q_{\frac{2(m-2)}{3}}, z_3 z_4, z_2 z_3 + z_4 z_5, \ldots, z_{k-3} z_{k-2}, z_{k-4} z_{k-3} + z_{k-2} z_{k-1}$ gen-
erates $I(G)$, up to radical. Hence we have $ara\, I(G) \leq \frac{2|V|-2}{3}$.

Case 2.4 Let $n \equiv m \equiv 0$.
In this case $|V| \equiv 0$. First assume that $n = m = 3$. We have that $I(G) = I \cap J$,
where $I = I(G) + (x_1 y_1 z_3 z_6 \ldots z_k)$ and $J = (x_2, x_3, y_2, y_3, z_1, z_2, z_4, z_5, \ldots, z_{k-5}, z_{k-4}, z_{k-2}, z_{k-1})$. Since J is a complete intersection ideal, we have that $cd\, J = 4 + \frac{2k}{3}$.
Moreover, $I + J = (x_1 y_1 z_3 z_6 \ldots z_k, x_2, x_3, y_2, y_3, z_1, z_2, z_4, z_5, \ldots, z_{k-5}, z_{k-4}, z_{k-2}, z_{k-1})$.
Since $I + J$ has grade equal to $5 + \frac{2k}{3}$, by [5, Theorem 6.2.7] we have $H^i_{I+J}(R) \neq 0$
in $i = 5 + \frac{2k}{3}$ and $H^i_{I+J}(R) = 0$ for any $i < 5 + \frac{2k}{3}$. In the Mayer-Vietoris sequence
for local cohomology (see [10], Section 3)

$$\ldots \to H^{4+\frac{2k}{3}}_{I+J}(R) \to H^{4+\frac{2k}{3}}_I(R) \oplus H^{4+\frac{2k}{3}}_J(R) \to H^{4+\frac{2k}{3}}_{I\cap J}(R) \to \ldots,$$

the left term is zero, whereas the middle term is not. It follows that the right term
is non zero, too. This implies that $pd\, I(G) = cd\, I(G) \geq 4 + \frac{2k}{3} = \frac{2|V|}{3}$.
So without loss of generality we may assume that $n > 3$. Then from Theorem C,
since $m - 3 \equiv 0$, we have that $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-6-\frac{2(k-3)}{3}-\frac{2(m-3)}{3}}(\varepsilon(L_{n-3}))$, for
all i. Hence, in view of Theorem D and (*), we have that $\tilde{H}_i(E_1 \cap E_2) \neq 0$ only
if $i = \frac{2|V|}{3} - 2$, in which case this homology group is K. In view of Theorem A,
Theorem B and (*) we also have that $\tilde{H}_i(E_1) \neq 0$ only if $i = \frac{2|V|}{3} - 2$, in which case
this homology group is K^2. The Mayer-Vietoris sequence shows that $\tilde{H}_i(\varepsilon(G)) \neq 0$
in $i = \frac{2|V|}{3} - 2$. Thus in view of (*) we deduce that $pd\, I(G) \geq \frac{2|V|}{3}$.
If $k = 0$, then, by Lemma 2.1, the sequence $B: x_1 y_1, q_0 + q'_0, q_1, \ldots, q_{\frac{2m}{3}-1}, q'_1, \ldots,$
$q'_{\frac{2n}{3}-1}$ of length $\frac{2|V|}{3}$ generates $I(G)$, up to radical, where the sequence $A_m: q_0, \ldots, q_{\frac{2m}{3}-1}$ generates $I(C_m)$, up to radical, and the sequence $A_n: q'_0, \ldots, q'_{\frac{2n}{3}-1}$
generates $I(C_n)$, up to radical, by Proposition 2.2.
If $k \geq 3$, then, by Lemma 2.1 the sequence $B: z_1 x_1, z_1 z_2 + q_0, q_1, \ldots, q_{\frac{2m}{3}-1}, z_3 z_4,$

$z_2 z_3 + z_4 z_5, \ldots, z_{k-3} z_{k-2}, z_{k-4} z_{k-3} + z_{k-2} z_{k-1}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2n}{3}-1}$ of length $\frac{2|V|}{3}$ generates $I(G)$, up to radical. We thus have $ara\, I(G) \leq \frac{2|V|}{3}$.

Case 3 Let $k \equiv 1$.
We can write $\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{y_1, y_2\}, \ldots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\},$
$\ldots, \{z_{k-1}, z_k\}, \{z_k, y_1\}; V) = E_1 \cup E_2$, where
$E_1 = \varepsilon(\{x_2, x_3\}, \ldots, \{x_m, x_1\}, \{y_1, y_2\}, \ldots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-1}, z_k\},$
$\{z_k, y_1\}; V)$ and $E_2 = \varepsilon(\{x_1, x_2\}; V)$. We have that $E_1 = \varepsilon(L_{m+k} \cup C_n)$, where
$L_{m+k} : x_2 \ldots x_m x_1 z_1 \ldots z_k$.
If $m = 3$, then, by [12, Lemma 3.4], $E_1 \cap E_2 = \varepsilon(\{x_3\}, \{z_1\}, \{z_2, z_3\}, \ldots, \{z_k, y_1\},$
$\{y_1, y_2\}, \ldots, \{y_n, y_1\}; V)$, so that, by [12, Lemma 3.5],

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-2}(\varepsilon(H_4)),$$

for all i, where H_4 is the induced subgraph on $V \setminus (V(C_m) \cup \{z_1\})$, i.e., the union of C_n and the path $L_k : z_2 \ldots y_1$. If $m \geq 4$, then $E_1 \cap E_2 = \varepsilon(\{x_3\}, \{x_m\}, \{z_1\}, \{x_4, x_5\},$
$\ldots, \{x_{m-2}, x_{m-1}\}, \{z_2, z_3\}, \ldots, \{z_k, y_1\}, \{y_1, y_2\}, \ldots, \{y_n, y_1\}; V)$, so that, by [12, Lemma 3.5],

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-3}(\varepsilon(L_{m-4} \cup H_4)),$$

for all i. Since E_2 is a simplex, $\tilde{H}_i(E_2) = 0$ for all i.

Case 3.1 Let $n \equiv 1$.
The sequence $A_n : q'_0, \ldots, q'_{\frac{2(n-1)}{3}}$ generates $I(C_n)$, up to radical, by Proposition 2.2.

Case 3.1.1 Let $m \equiv 0$ or $m \equiv 2$.
First let $m \equiv 0$. Then $|V| \equiv 2$. If we apply Corollary 3.2 to the path $L_{m+k-1} :$
$x_2 \ldots x_m x_1 \ldots z_{k-1}$, and then Lemma 3.1 for $v = z_k$ we get, that, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m+k-1)}{3}-3}(\varepsilon(L_{n-3}))$, which is zero for all i by Theorem D and (*). If $m \geq 6$, applying Theorem C ($m - 4 \equiv 2$) and Corollary 3.2 to $E_1 \cap E_2$ along the path $L_{k-1} : z_2 \ldots z_k$, we deduce that, for all i,

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-\frac{2(m-4)-1}{3}-\frac{2(k-1)}{3}-3}(\varepsilon(C_n)),$$

which is also true for $m = 3$.
By Theorem B and (*), $\tilde{H}_{\frac{2|V|+2}{3}-3}(E_1 \cap E_2) \neq 0$. So by the Mayer-Vietoris sequence $\tilde{H}_{\frac{2|V|+2}{3}-2}(\varepsilon(G)) \neq 0$ and in view of (*) we conclude that $pd\, I(G) \geq \frac{2|V|+2}{3}$.
By Lemma 2.1, $B : q'_0, \ldots, q'_{\frac{2(n-1)}{3}}, x_1 z_1, z_1 z_2 + q_0, q_1, \ldots, q_{\frac{2m}{3}-1}, z_3 z_4, z_2 z_3 + z_4 z_5,$
$\ldots, z_{k-1} z_k, z_{k-2} z_{k-1} + z_k y_1$ of length $\frac{2|V|+2}{3}$ generates $I(G)$, up to radical, where sequence $A_m : q_0, \ldots, q_{\frac{2m}{3}-1}$ generates $I(C_m)$, up to radical, by Proposition 2.2.
Therefore, we have $ara\, I(G) \leq \frac{2|V|+2}{3}$.
Now let $m \equiv 2$. In this case $|V| \equiv 1$, and $m + k \equiv 0$. Moreover, by Theorem C, since $m - 4 \equiv 1$, we have that $\tilde{H}_i(E_1 \cap E_2) = 0$ for all i. Hence $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_i(E_1)$ for all i. Applying Corollary 3.2 to the path $L_{m+k} : x_2 x_3 \ldots x_1 z_1 \ldots z_k$ we obtain that, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m+k)}{3}}(\varepsilon(C_n))$. By Theorem B and (*), $\tilde{H}_i(E_1) \neq 0$ in $i = \frac{2|V|+1}{3} - 2$. Thus by (*) we have $pd\, I(G) \geq \frac{2|V|+1}{3}$.
By Lemma 2.1, the sequence $B : x_1 z_1, q_0 + z_1 z_2, q_1, \ldots, q_{\frac{2(m-2)}{3}}, z_3 z_4, z_2 z_3 + z_4 z_5, \ldots, z_{k-1} z_k, z_{k-2} z_{k-1}+$

$z_k y_1, q'_0, \ldots, q'_{\frac{2(n-1)}{3}}$ of length $\frac{2|V|+1}{3}$ generates $I(G)$, up to radical, where the sequence $A_m : q'_0, \ldots, q'_{\frac{2(m-2)}{3}}$ generates $I(C_m)$, up to radical, by Proposition 2.4. Thus $ara\, I(G) \leq \frac{2|V|+1}{3}$.

Case 3.1.2 Let $m \equiv 1$.
In this case $|V| \equiv 0$. Consider the induced subgraph H_5 on $V \setminus \{z_k\}$. We have, for all i, $\tilde{H}_i(\varepsilon(H_5)) = \tilde{H}_i(H''' \cup C_n)$, where H''' is the induced subgraph on $V \setminus (V(C_n) \cup \{z_k\})$, i.e., the union of C_m and the path $L_k : x_1 z_1 \ldots z_{k-1}$. Applying Theorem A and then Corollary 3.2 to H''' along the path $L_{k-1} : z_1 \ldots z_{k-1}$ we have $\tilde{H}_i(\varepsilon(H_5)) = \tilde{H}_{i - \frac{2n+1}{3} - \frac{2(k-1)}{3}}(C_m)$, for all i, and this homology group, by Theorem B and (*), is non zero in $i = \frac{2|V|}{3} - 2$. So $\tilde{H}_{\frac{2|V|}{3} - 2}(\varepsilon(H_5)) \neq 0$. In view of (*) we deduce that $pd\, I(G) \geq \frac{2|V|}{3}$.
The sequence $B : q_0, q_1, q_{\frac{2(m-1)}{3}} + z_1 x_1, \ldots, q_{\frac{2(m-1)}{3} - 1}, , q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + z_k y_1, \ldots,$

$q'_{\frac{2(n-1)}{3} - 1}, z_2 z_3, z_1 z_2 + z_3 z_4, \ldots, z_{k-2} z_{k-1}, z_{k-3} z_{k-2} + z_{k-1} z_k$ of length $\frac{2|V|}{3}$, generates $I(G)$, up to radical, by Lemma 2.1, where the sequence $A_m : q_0, \ldots, q_{\frac{2(m-1)}{3}}$ generates $I(C_m)$, up to radical, by Proposition 2.3. Therefore, we have that $ara\, I(G) \leq \frac{2|V|}{3}$.

Case 3.2 Let $m \equiv 2$, and $n \equiv 0$ or 2.
In this case $m + k \equiv 0$. Applying Corollary 3.2 to the path $L_{m+k} : x_2 x_3 \ldots x_1 z_1 \ldots z_k$ we obtain that, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i - \frac{2(m+k)}{3}}(\varepsilon(C_n))$.
Moreover, the sequence $A_m : q_0, \ldots, q_{\frac{2(m-2)}{3}}$ generates $I(C_m)$, up to radical, by Proposition 2.4.
First let $n \equiv 0$. Then $|V| \equiv 0$ and, by Theorem B and (*), $\tilde{H}_i(E_1) \neq 0$ in $i = \frac{2|V|}{3} - 2$. Thus by (*) we have $pd\, I(G) \geq \frac{2|V|}{3}$.
By Lemma 2.1, the sequence $B : x_1 z_1, q_0 + z_1 z_2, q_1, \ldots, q_{\frac{2(m-2)}{3}}, z_3 z_4, z_2 z_3 + z_4 z_5, \ldots, z_{k-1} z_k, z_{k-2} z_{k-1} + z_k y_1, q'_0, q'_1, \ldots, q'_{\frac{2n}{3} - 1}$ of length $\frac{2|V|}{3}$ generates $I(G)$, up to radical, where the sequence $A_n : q'_0, \ldots, q'_{\frac{2n}{3} - 1}$ generates $I(C_n)$, up to radical by Proposition 2.2. Thus $ara\, I(G) \leq \frac{2|V|}{3}$.
If $n \equiv 2$, then $|V| \equiv 2$ and, by Theorem B and (*), $\tilde{H}_i(E_1) \neq 0$ in $i = \frac{2|V|-1}{3} - 2$. Thus by (*) we have $pd\, I(G) \geq \frac{2|V|-1}{3}$.
By Lemma 2.1, the sequence $B : q_0, \ldots, q_{\frac{2(m-2)}{3}}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_1 z_2, x_1 z_1 + z_2 z_3, \ldots, z_{k-3} z_{k-2}, z_{k-4} z_{k-3} + z_{k-2} z_{k-1}$ of length $\frac{2|V|-1}{3}$ generates $I(G)$, up to radical, where the sequence $A_n : q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$ generates $I(C_n)$, up to radical by Proposition 2.4. Thus $ara\, I(G) \leq \frac{2|V|-1}{3}$.

Case 3.3 Let $n \equiv m \equiv 0$.
In this case $|V| \equiv 1$. As in Case 1, we can write $\varepsilon(G) = E_1 \cup E_2$, where $E_1 = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{y_1, y_2\}, \ldots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-1}, z_k\}; V)$ and $E_2 = \varepsilon(\{z_k, y_1\}; V)$.
Applying Corollary 3.2 to the path $L_{k-1} : x_1 z_1 \ldots z_{k-2}$, we have that, for all i,

$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-\frac{2(k-1)}{3}-3}(\varepsilon(L_{m-1} \cup L_{n-3}))$. Theorem C ($m-1 \equiv 2$), Theorem D and (*) show that $\tilde{H}_i(E_1 \cap E_2) \neq 0$ only if $i = \frac{2|V|+1}{3} - 3$. Since E_2 is a simplex, $\tilde{H}_i(E_2) = 0$ for all i. Applying Corollary 3.2 to E_1 along the path $L_{k-1} : z_2 \ldots z_k$, and once again Lemma 3.1 for $v = z_1$, we obtain that, for all i, $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(k-1)}{3}-3}(\varepsilon(L_{m-3} \cup C_n))$, which by Theorem C, Theorem B and (*), is non zero only in $i = \frac{2|V|+1}{3} - 2$. The Mayer- Vietoris sequence shows that $\tilde{H}_i(\varepsilon(G)) \neq 0$ in $i = \frac{2|V|+1}{3} - 2$. Thus, in view of (*), we have that $pd\, I(G) \geq \frac{2|V|+1}{3}$. By Lemma 2.1, the sequence $B: x_1z_1, z_1z_2+q_0, q_1, \ldots, q_{\frac{2m}{3}-1}, z_3z_4, z_2z_3+z_4z_5, \ldots,$ $z_{k-1}z_k, z_{k-2}z_{k-1} + z_ky_1, q'_0, \ldots, q'_{\frac{2n}{3}-1}$, generates $I(G)$, up to radical, where the sequence $A_m: q_0, \ldots, q_{\frac{2m}{3}-1}$ generates $I(C_m)$, up to radical, and the sequence $A_n: q'_0, \ldots, q'_{\frac{2n}{3}-1}$ generates $I(C_n)$, up to radical, by Proposition 2.2. This implies that $pd\, I(G) = ara\, I(G) = \frac{2|V|+1}{3}$ in this case. This completes the proof. $\square$

From Theorem 3.3 and Theorem 3.4 we deduce the following corollary.

Corollary 3.5. *Let G be a bicyclic graph, then $ara\, I(C_n) = pd\, I(C_n)$.*

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